Oscillation in Price – Adjustment Models

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Abstract

The aim of this paper is to offer the supply and demand model describing how price vary as a result of a balance between supply and demand. We present two models in which price tends to equilibrium point not only monotonically but can also oscillates around the equilibrium point.

Keywords: Walrasian assumption, differential equations, oscillation

1. Introduction

It is known from microeconomics, that in perfectly competitive market, equilibrium is determined by the point at which the supply function and the demand function are equal. Here we limit ourselves to the simple case in which demand and supply of a commodity are assumed to depend solely on the price of that commodity.

As usual, supply is the amount of commodity which is offered by the producer at each price. We shall use the notation $S(p)$, where p is price of commodity at time t. Demand is the amount of commodity which are people willing to buy at a certain price, i.e. the term demand signifies the ability or the willingness to buy a particular commodity at a given point of a time. We shall use the notation $D(p)$.

The relationship of price and demand can be exhibited graphically as the demand curve. This curve in generally negatively sloped - decreasing in mathematical term. On the other hand, supply curve is the correlation between price and demand. Supply curve is positively sloped - increasing. According to the Walrasian assumption, price tends to increase (decrease) if demand is greater than (less) than supply.

The general dynamic formalization of the Walrasian assumption is the following:

$$p^{'} = f(D(p) - S(p)),$$

where it is assumed that $xf(x) > 0$ for $x \neq 0$. We shall study properties of this model using linear approximation of all function as follows $f(x) = k_1 x,\ k_1 > 0$ and

$$D\big(p(t)\big) = d_1 + d_2\, p(t), \qquad d_2 < 0, \tag{1.1}$$
$$S\big(p(t)\big) = s_1 + s_2\, p(t), \qquad s_2 > 0,$$

The assumptions $d_2 < 0$, and $s_2 > 0$ reflect the fact that demand and supply curve are negatively and positively sloped, respectively.

Applying this linearization, the equation for the rate of change of price takes the form

$$p^{'}(t) = k_1\big(D(p) - S(p)\big). \tag{E_1}$$

Employing (1.1), we have

$$p^{'}(t) = k_1\big(d_1 - s_1 + (d_2 - s_2)p(t)\big).$$

Using the notation $-K = k_1(d_2 - s_2) > 0$ and $p_e = \frac{d_1 - s_1}{s_2 - d_2} > 0$, this equation simplifies to

$$p^{'}(t) = -K(p(t) - p_e)$$

Coefficient p_e is equilibrium price, for $p(t) = p_e$ there is no change of price. It is easy to see that p_e is a point where $D(p) = S(p)$. Considering deviation from the equilibrium $z(t) = p(t) - p_e$ and observing that $z^{'}(t) = p't$, we see that

$$z^{'}(t) = -Kz(t).$$

This is simple linear differential equation with general solution $z(t) = ce^{-Kt}$. Therefore, in term of price we have $p(t) = p_e + ce^{-Kt}$. Denoting the initial value $p(0) = p_0$, we obtain the final expression for price in the form

$$p(t) = p_e + (p_0 - p_e)e^{-Kt}.$$

From this formula we conclude that no matter how p_0 is chosen, price $p(t)$ always tends monotonically to its equilibrium point p_e, i.e. equilibrium point is stable.

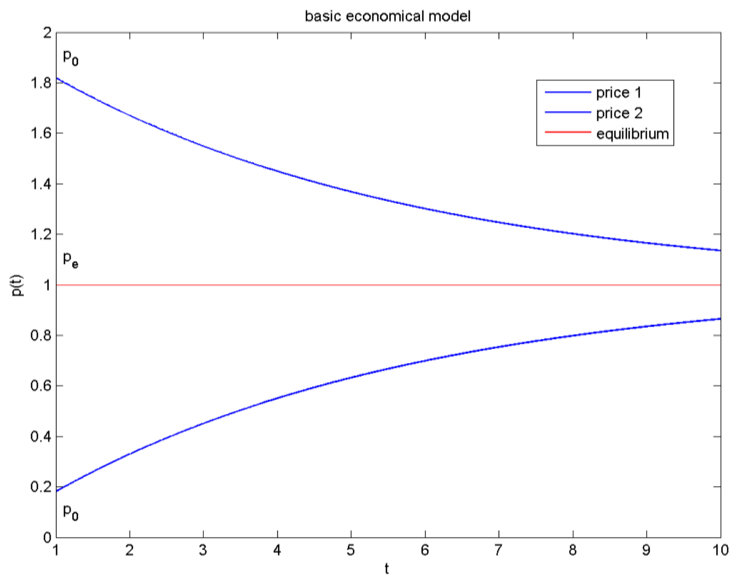

Figure 1: Monotonic behavior of price

Our aim in this paper is to suggest two models in which price can oscillate around its equilibrium point.

2. *Main Results*

2.1 Price – adjustment model with supply delay

We consider demand and supply model described by differential equation (E_1) under assumption that supply responds to vary of price with certain delay. This situation can be mathematically formalized as

$$S\big(p(t)\big) = s_1 + s_2\, p(t - \tau).$$

The delay τ expresses time needed for realization of change of supply in dependence on trend of price. In other words, present production depends on past price (past decision).

Setting delay supply into (E_1), we obtain

$$(E_2) \qquad p^{'}(t) = k_1\big(d_1 - s_1 + d_2 p(t) - s_2 p(t - \tau)\big).$$

This is delay differential equation and behavior of its solutions is different from that of equation without deviating argument.

The main questions we are asking are the following:

- What is price equilibrium point?
- Will price tend to move towards its equilibrium?
- If yes, is this convergent monotone or not?
- Is the equilibrium point stable?

Before giving answer we do some mathematical calculation to transform (E_2) into simpler form. Setting
$a = k_1(d_1 - s_1)$, with $d_1 > s_1$; $b = -k_1 d_2$; $c = k_1 s_2$;
we see that

$$p'(t) - a + bp(t) + cp(t - \tau) = 0,$$

which in view of definition of p_e and the transformation $z(t) = b(p(t) - p_e)$ takes the form

$$z'(t) + bz(t) + cz(t - \tau) = 0.$$

Applying one more substitution $v(t) = z(t)e^{bt}$, one gets
(2.1) $$v'(t) + ce^{bt}\mathrm{v}(t - \tau) = 0.$$

This is delay differential equation with constant coefficients and we recall from (Ladde & Lakshmikantham & Zhang, 1987) some its properties.

If the following conditions

$$\frac{\pi}{2} > c\tau e^{bt} > \frac{1}{e}$$

hold, then every solution $v(t)$ of (2.1)

a) oscillates around zero,

b) tends to zero as $t \to \infty$.

On the other hand, used double transformation, namely

$$b(p(t) - p_e) = z(t) = v(t)e^{-bt}$$

provides the relationship between properties of solutions $v(t)$ of of (2.1) and $p(t)$ of (E_2), consequently we can deduce the following properties of (E_2)

Assuming that

$$\frac{\pi}{2} > k_1 s_2 \tau e^{-k_1 d_2 \tau} > \frac{1}{e}$$

hold, we conclude that every solution $p(t)$ of (E_2)

a) oscillates around equilibrium point p_e,

b) tends to equilibrium point p_e as $t \to \infty$,

c) the equilibrium point p_e is (asymptotically) stable.

Thus delay in supply function may cause oscillation of price around its equilibrium point, (see Fig.2) which is new phenomena that cannot occur in economical models based on differential equations without delay. On the other hand, there exist also solutions that monotonically tend to equilibrium point p_e provided that

$$k_1 s_2 \tau e^{-k_1 d_2 \tau} \le \frac{1}{e}$$

So our model yields more variability for behavior of price.

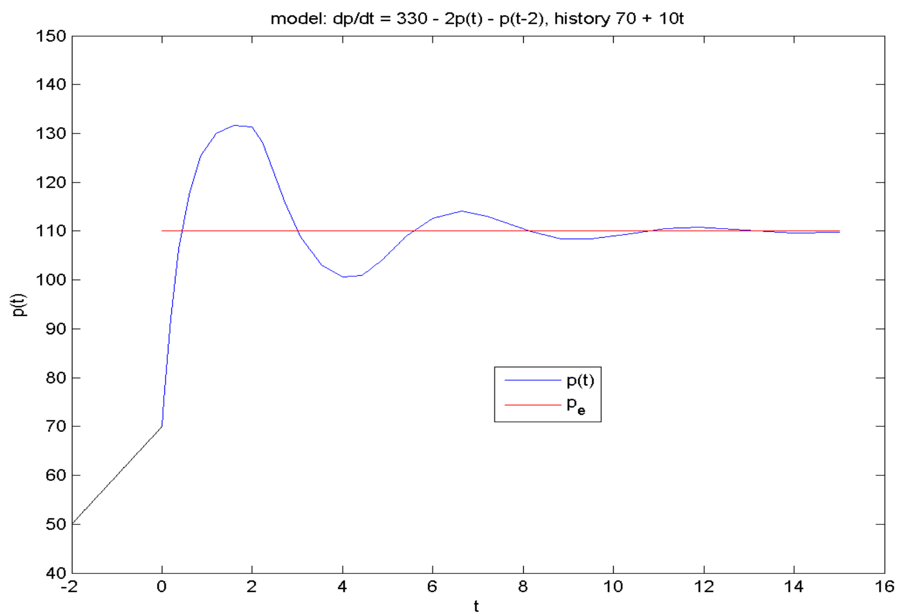

Figure 2: Oscillation of price around equilibrium

2.2 Price – adjustment model with inventories

In the previous section we studied dynamics of price adjustments in a model of a competitive market. We have supposed that price adjusts reflects the difference between demand and supply as follows

(E_1) $$p'(t) = k_1\big(D(p) - S(p)\big), \quad k_1 > 0.$$

However, this model neglects the inventory of unsold merchandize.

In the sequel, we shall study how the dynamics of price adjustments will be affected if we take into account this inventory. It is natural to assume that inventory has negative effect on the price. Mathematical formulation of this consideration is following

(E_3) $$p'(t) = k_1\big(D(p) - S(p)\big) - k_2 \int_0^t \big(S(u) - D(u)\big)\,\mathrm{d}u$$

with $k_1 > 0,\ k_2 > 0$. The second term expresses accumulated stock as the integral of past differences. With $k_2 > 0$ this term really causes price to adjust downward when the inventory is positive.

The obtained price adjustment model with inventories is described by integro-differential equation. Let us explore properties of solutions (price) of this equation. Differentiation of it, simplifies this equation into the form

$$p''(t) = k_1\big(D'(p) - S'(p)\big) - k_2\big(S(p) - D(p)\big).$$

Setting linear approximation of $D(p)$ and $S(p)$ of (1.1), we obtain the second order linear differential equation with constant coefficients

$$p''(t) + \alpha p'(t) + \beta(p(t) - p_e) = 0,$$

where

$$\alpha = k_1(s_2 - d_2) > 0, \quad \beta = k_2(s_2 - d_2) > 0; \qquad p_e = \frac{d_1 - s_1}{s_2 - d_2} > 0.$$

Using substitution $z(t) = p(t) - p_e$, we obtain the equation

$$z''(t) + \alpha z'(t) + \beta z(t) = 0.$$

It is known from theory of differential equations that either all solutions of this equation are oscillatory (i.e., they oscillate around zero) or monotonic. Solving this equation – via characteristic equation – we obtain in notation $r_{1,2} = \left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right)/2$ the following formulas for general solution:

a) if $\alpha^2 > 4\beta$, then

$$z(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

b) if $\alpha^2 = 4\beta$, then

$$z(t) = c_1 e^{-\frac{\alpha}{2}t} + c_2 t e^{-\frac{\alpha}{2}t},$$

c) if $\alpha^2 < 4\beta$, then

$$z(t) = c_1 e^{-\frac{\alpha}{2}t} \cos\left(\frac{\sqrt{4\beta - \alpha^2}}{2}\ t\right) + c_2 e^{-\frac{\alpha}{2}t} \sin\left(\frac{\sqrt{4\beta - \alpha^2}}{2}\ t\right),$$

where c_1 and c_2 are constants. Since $z(t) = p(t) - p_e$, we can immediately obtain the corresponding results for behavior of price $p(t)$ and the following results provide complete analysis of Price – adjustment model with inventories:

a) if $\alpha^2 > 4\beta$, then the general solution of (E_3)

$$p(t) = p_e + c_1 e^{\left[-\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4\beta}}{2}\right]t} + c_2 e^{\left[-\frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 4\beta}}{2}\right]t}$$

moreover, price monotonically tends to equilibrium point p_e,

b) if $\alpha^2 = 4\beta$, then the general solution of (E_3)

$$p(t) = p_e + c_1 e^{-\frac{\alpha}{2}t} + c_2 t e^{-\frac{\alpha}{2}t},$$

what is more, price monotonically tends to equilibrium point p_e,

c) if $\alpha^2 < 4\beta$, then the general solution of (E_3)

$$p(t) = p_e + c_1 e^{-\frac{\alpha}{2}t} \cos\left(\frac{\sqrt{4\beta - \alpha^2}}{2}\ t\right) + c_2 e^{-\frac{\alpha}{2}t} \sin\left(\frac{\sqrt{4\beta - \alpha^2}}{2}\ t\right),$$

in addition, price oscillates around to equilibrium point p_e and tends to p_e.

While for the price - adjustment model with supply delay the oscillation of price around equilibrium point was caused by delay argument of differential equation for price - adjustment model with inventories oscillation follows from the order of differential equations

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