Pre-service Mathematics Teachers' Perception Levels Concerning the Concepts of Derivative and Differential

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Abstract

This study investigates pre-service mathematics teachers' perception levels concerning the concepts of derivative and differential and the difference between these concepts. The study sample consists of the pre-service secondary mathematics teachers in the K.K. Faculty of Education at Ataturk University. 76 individuals were selected as the sample group. As the data collection instrument, the study used five open-ended questions prepared by the researchers. The obtained data revealed that pre-service secondary mathematics teachers failed to thoroughly comprehend the differences between the concepts of derivative and differential. In other words, it was found that they interchangeably used the concepts of derivative and differential.

Keywords: Derivative, differential, learning difficulty

1. Introduction

Among the major problems in mathematics instruction is the teaching and learning of the basic concepts**.** Therefore, teachers should take greater care to make their students comprehend these basic concepts. Derivative and differential are two of such concepts with a critical role in analysis courses. One of the common issues in mathematics, engineering, physics, economics, chemistry and statistics is the limiting case of the ratio of the change that an increase in a variable creates in the function to the increase in the variable. This is used to explain the slope of the tangent in mathematics, speed and velocity in physics, reaction rate in chemistry, and the concepts of marginal revenue and marginal price in economics (Balcı, 2008). This concept is referred to as derivative in mathematics. Two problem cases led to the invention of the concept of derivative, which are calculating the instantaneous velocity of a moving particle and drawing a tangent to a curve on a point on the curve (Kadıoğlu ve Kamali, 2005). Of these, the first problem is a physics problem while the second is a geometrical problem.

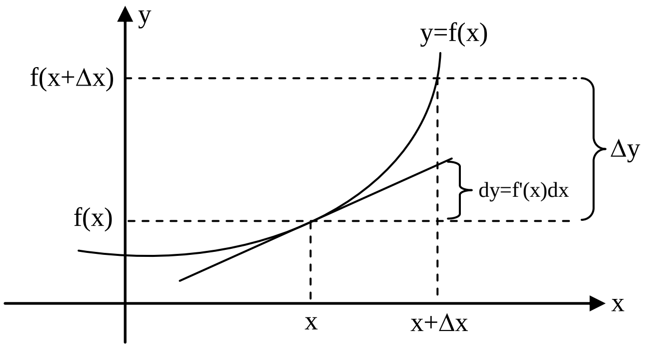
As one of the greatest inventions in mathematics, the historical process of derivative began in the 1600s, assuming its present form with the foundations of analysis. Grabnier (1983), a historian of mathematics, summarizes this process as follows: "The derivative was first used; it was then discovered; and it was finally defined". Consequently, the derivative (as a tangent) was used by Fermat and others in the mid-17th century; was then discovered by Newton and Leibniz towards the end of the same century; and was explored and developed rapidly during the 18th century. Its definition finally came in the 19th century. The derivative was defined by Lagrange in 1790 (the algebraic definition), by Cauchy in 1820 (the notions of limit and infinitesimals) and subsequently by Weierstrass in 1870 (with the concepts of epsilon and delta). Since the concept of limit had not yet been discovered by the time the concept of derivative was already known, derivative was defined with reference to the concept "infinitesimal" instead of the concept of limit. Only later, after the discovery of the limit concept, the derivative was defined using the limit. The derivative of a function at a point could be defined as follows: Let f be a function defined in the open interval (a, b) and $x_o, \in (a,b)$.
If the limit $\lim\limits_{h\to o}\dfrac{f(x_o+h)-f(x_o)}{h}$ exists, then function f is said to be differentiated at point x_0 and is denoted by $f'(x_o)=\lim\limits_{h\to o}\dfrac{f(x_o+h)-f(x_o)}{h}$. The value $f'(x_0)$ is referred to as the derivative of function f at point x_0. In this statement, if $x = x_0 + h$ is taken, then

$f'(x_o)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_o)}{x-x_o}$ and if Δx is taken instead of h, then

$$f'(x_o)=\lim_{\Delta x\to O}\frac{f(x_0+\Delta x)-f(x_o)}{\Delta x}.$$

The derivative is defined as the limit of fraction $\frac{\Delta y}{\Delta x}$ for $\Delta x \to 0$ and this limit is represented by $\frac{dy}{dx}$ to avoid forgetting the division symbol. So it is written as $\frac{dy}{dx} = \lim_{\Delta x \to O} \frac{\Delta y}{\Delta x}$. Here, $\frac{dy}{dx}$ should never be considered as the division of dy by dx. As mentioned above, the derivative has two meanings, one of which is its geometrical meaning. The slope of a tangent drawn to a curve at a point on the curve is equal to the derivative value of the function at that point. Thus, the slope is defined as follows;

$\tan\alpha = m = \lim_{x \to x_0} m(x) = f'(x_o) = \lim_{\Delta x \to O} \frac{f(x_0 + \Delta x) - f(x_o)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_o)}{x - x_o}$. One of the important concepts of analysis is the concept of differential. The concept of differential was defined using the concept of "infinitesimal": Let $y = f(x)$ be a function differentiable in an open interval containing x. The differential of x is denoted by dx and the statement $dy = f'(x)dx$ is called the differential of y. The following graph attempts to demonstrate the geometrical meaning of the differential:



As seen in the graph, increment in $\Delta y = f(x+\Delta x) - f(x)$ is the change in y from x to $x+\Delta x$ along the slope $y = f(x)$, while dy, the differential of y, is the change in y from x to $x+\Delta x$ along the tangent line. As is clear from this explanation, Δx and dx are different symbols for the same magnitude (in applications, if Δx is very close to zero, it is denoted by dx), while dy and Δy are entirely different. Moreover, as is apparent in the graph, the ratio of the change in the ordinate to the change in the abscissa in function $y = f(x)$ also gives the derivative. The slope of the tangent of the curve at point (x, y) is $\frac{dy}{dx}$. Leibniz referred to this as "differential" quotient (Kleiner 2001).

Accordingly, the difference between derivative and differential could be explained as follows: in function $y = f(x)$, we will investigate change Δy=f(x+Δx)-f(x) that occurs in dependent variable y when independent variable x is changed by Δx. If $y = f(x)$ is a function differentiable in any interval, then we can write

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$. Since the limit of $\frac{\Delta y}{\Delta x}$ is approximated to a value such as $f'(x)$, ratio $\frac{\Delta y}{\Delta x}$ is infinitesimally different from derivative $f'(x)$ compared to Δx. Thus, we can write down $\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon ...(1)$, where function $\varepsilon = \varepsilon(\Delta x)$ is an infinitesimal that meets the condition $\lim_{\Delta x \to 0} \varepsilon = 0$. Because in case x → a or x → ∞, if lim f(x) = m and k is an infinitesimal, then $f(x) = m + k$. If both sides of (1) are multiplied by Δx, then we obtain $\Delta y = f'(x)\Delta x + \varepsilon \Delta x$. As seen in the last statement, as opposed to increment Δx in the independent variable of the function, increment Δy is composed of two parts. The first part is $f'(x)\Delta x$, while the second part $\varepsilon \Delta x$, Δx multiplied by ε, which is an infinitesimal close to zero. Statement $f'(x)\Delta x$ is called the differential of function $y = f(x)$ and is represented by $dy = df(x) = f'(x)\Delta x$ …(2).

Taking $y = f(x) = x$, $dy = dx = x^{'}\Delta x = \Delta x$ or briefly $dx = \Delta x$. Then, the differential and increment of the free variable are equal. Thus, we can write formula (2) as $dy = f^{'}(x)dx$. As a conclusion, it could be argued that the differential of a function at any point is equal to the derivative of the function at this point multiplied by the increment of its variable. In other words, the differential is the derivative multiplied by dx.

Many mathematics teachers use the concepts of derivative and differential interchangeably. These two concepts are in fact different although they appear to be similar. If the difference between the two concepts is not clearly understood, students will inevitably experience learning difficulties particularly in Analysis courses. There are many studies on understanding the concept of derivative (Orton 1983b; Heid 1988; Tufte 1989; Amit and Vinner 1990; Ferrini-Mundy and Graham 1994; Kowalezky and Hausknecht 1994; Schwabach and Dosemagen 2000; Aspinwal and Miller 2001, Berry and Nyman 2003). Zhang (2003) argues that teaching students the basic theorems and concepts of analysis is to make them comprehend mathematical symbols and concepts; to improve their ability to think creatively, deeply, logically, and intellectually and their capacity to imagine and calculate; and to help them use the skills they acquire in their future lives. Tall (1993) demonstrated that students have conceptual difficulty about whether expression $\frac{dy}{dx}$ should be perceived as a single representational symbol or as a fraction.

In the same study, the author also discussed whether terms du in equation $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ can be simplified. In a study, Ubuz (1996) asked questions such as "what does derivative mean" and "what does $\frac{\Delta y}{\Delta x}$ mean", which the students replied, respectively, as: "derivative is small change" and "$\frac{\Delta y}{\Delta x}$ is a small change in slope". However, $\frac{\Delta y}{\Delta x}$ should be perceived as "$\frac{\Delta y}{\Delta x} = \frac{y(x+h) - y(x)}{x+h-x} = \frac{small\, change\, in\, y}{small\, change\, in\, x}$" or "slope of a *secant* line". From all these discussions, we conclude that the difference between derivative and differential is not adequately understood. Identification of the learning difficulties regarding these two concepts may help perceiving better the difference between the concepts of derivative and differential. Thus, this study aims to identify the differences between the concepts of derivative and differential.

2. Methodology

2.1. Problem

This study's problem relates to the extent to which the difference between the concepts of derivative and differential is understood by fifth-year students at the department of secondary mathematics teacher training.

2.2. Purpose

Mathematical subjects have a more sequential character when compared to the subjects in other courses. This is mainly because mathematics produced itself without any external contributions. No concept can be properly taught before other prerequisite concepts are thoroughly learned (Altun, 2005). Therefore, it is of critical importance that certain concepts become meaningful in students' minds. For learners of advanced mathematics, two of such concepts are the concepts of derivative and differential. These concepts play a crucial role in Analysis courses. Nevertheless, the subtle difference between the two concepts often remains insufficiently understood even by instructors. In this context, the purpose of the present study is to investigate the levels of perception concerning the concepts of derivative and differential among the students of the department of secondary mathematics teacher training, who will be appointed as teachers in a year's time.

2.3. Material and Method

In this study which aims to determine the extent to which the difference between the concepts of derivative and differential is understood by fifth-year students at the department of secondary mathematics teacher training, the "case study design" was selected as the most appropriate research model for the study subject since there was a need to collect more extensive information about the subject (Yildirim and Simsek 2000; Mcmillan and Schumacher 2001).

2.4. Sampling

The study sample consists of the students in the Department of Secondary Mathematics Teacher Training in Kazim Karabekir Faculty of Education at Ataturk University. 76 students were selected as the sample group.

2.5. Data Collection Instruments

The data were collected through an information test containing open-ended questions formulated by the researchers to determine whether the difference between derivative and differential is understood. When administering the information test, the pre-service mathematics teachers were allocated sufficient time without any concerns about time.

3. Findings

The responses of the pre-service teachers in the sample group to the open-questions in the information test were examined in detail and are presented in separate tables for each question. As known, one of the applications of the differential concept is finding an approximate value. With regard to this application, the instructor asked the pre-service teachers in question 1 to find the approximate value for $\sqrt{80}$. The responses provided are presented in the table below.

Table 1. Responses to question 1 (frequencies) (N=76).

Response categories	Frequencies
Attempting to the find the value by extracting the approximate value of $\sqrt{80}$ Stating that $8 < \sqrt{80} < 9$	46
Finding the value by associating with the differential concept	15
Stating that it will expand to Taylor series and computed with a certain margin of error	4
Attempting to find it using the mean value theorem	3
No response	8

In the second open-ended question in the information test, the students were asked to explain whether expression $\frac{dy}{dx}$ denotes a fraction by stating their reasons. The responses to this question are shown in the following table.

Table 2. Responses to question 2 (frequencies) (N=76).

Response categories	Frequencies
It does not denote a fraction (no explanation)	17
It denotes a fraction (no explanation)	12
The given expression is a derivative of independent variable x versus dependent variable y or the terms dy and dx are derivative operators or operands and thus the given expression cannot denote a fraction	19
If the denominator in $\frac{dy}{dx}$ is $\left(dx \neq 0\right)$, it does not denote a fraction	3
No certain proportioning can be made between the terms dy and dx, dy is not a whole and thus cannot be broken into its parts (fraction is defined as breaking a whole into parts); accordingly it does not denote a fraction	3
Since $dy, dx \notin Z$ it does not denote a fraction	1
By providing an example, stating that expression $\frac{dy}{dx}$ denotes a fraction if its result is a number of function	10

Since it contains a numerator, a denominator and a fraction line, expression $\frac{dy}{dx}$ denotes a fraction	2
No response	9

The third question in the information test asked the respondents to explain whether the terms du and dz in expression $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dz} \cdot \frac{dz}{dx}$ can be simplified by stating the reasons. The responses to the question are presented in the following table.

Table 3. Responses to question 3 (frequencies) (N=76).

Response categories	Frequencies
They can be simplified (no explanation)	16
They cannot be simplified (no explanation)	17
They cannot be simplified because the expressions do not denote a fraction and it is not a division operation	4
They cannot be simplified because the terms du and dz denote a derivative or differential	18
They cannot be simplified because the expressions in the question are dependent and independent variables in relation to each other	5
They can be simplified because the expressions denote a fraction and They are multiplied	4
They can be simplified because the result is found by calculating the derivatives of the expressions in relation to each other (by providing an example)	7
They can be simplified because du signifies calculating the derivative, while $\frac{1}{du}$ denotes calculating the integral	1
No response	4

The fourth open-ended question in the information test asked the respondents to explain whether equation $\frac{dy}{dx} = \frac{\Delta y}{\Delta x}$ is correct by stating the reasons. The responses to this question are shown in the table below.

Table 4. Responses to question 4 (frequencies) (N=76).

Response categories	Frequencies
It is correct (no explanation)	11
It is incorrect (no explanation)	30
It is incorrect because $\frac{\Delta y}{\Delta x}$ is the amount of increase and $\frac{dy}{dx}$ is the derivative	6
It is incorrect because Δy and Δx are the amount of increase and dy and dx are differential	3
It is incorrect because dy denotes the derivative of y and Δy denotes a function	2
It is incorrect because one denotes the derivative and the other denotes the differential (but which is which is not explained)	3
It is incorrect because $\frac{dy}{dx}$ is the derivative and $\frac{\Delta y}{\Delta x}$ is the differential	2

It is incorrect because $\frac{dy}{dx}$ is the derivative and $\frac{\Delta y}{\Delta x}$ is the integral	1
It is correct because both denote the derivative	3
It is correct because both signify a change	2
No response	13

The fifth open-ended question in the information test asked the students to explain whether the derivative and differential are the same concepts by stating the reasons. The responses to the question are shown in the following table.

Table 5. Responses to question 5 (frequencies) (N=76).

Response categories	Frequencies
They are the same concepts (no explanation)	9
Derivative is the slope of the tangent lines at all points, while differential is the slope of the tangent lines at all points .	7
Derivative denotes a point, while differential is linear	4
These two concepts are the opposite of each other	3
$\frac{dy}{dx} = f^{'}(x)$ is the derivative, $dy = f^{'}(x)dx$ is the differential	10
The derivative is the result or the product, while the differential is an operation	3
The derivative is more general than the differential or vice versa	4
$\frac{dy}{dx}$ is the derivative, while dy is the differential	5
The derivative is an operation, while the differential is a symbol	11
The derivative means taking derivative once for a single variable, while the differential means taking derivative once for multiple variables	4
The derivative denotes a point, while the differential denotes a ratio	7
No response	9

4. Conclusion and Implications

An evaluation of the obtained data revealed that the pre-service secondary mathematics teachers fail to fully comprehend the difference between the concepts of derivative and differential. In other words, the findings demonstrated that the pre-service teachers often perceive the concepts of derivative and differential as the same. Since the concepts of derivative and differential play an important role in learning calculus, it is crucial for future learnings that students can properly make sense of these concepts. Therefore, it is obvious that pre-service teachers should be taught the difference between the concepts of derivative and differential. Furthermore, the concepts of derivability and differentiability in analysis are used interchangeably, which creates the misconception in students' minds that the concepts of derivative and differential can also be used interchangeably. Instructors who teach these concepts need to make clear for their students the distinction between these two concepts. A similar study could be conducted with in-service secondary mathematics teachers to determine the teachers' levels of perception concerning the difference between the concepts of derivative and differential.

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