Modeling non-linear Behavior of Independent Variables

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Abstract

The modeling non-linear behavior between independent variables and response variable remained a challenging task for the researchers. We consider precipitation data set for two monitoring stations for twenty seven years period during monsoon. Generalized linear models, Generalized additive model and piecewise regression models are used to find appropriate model to describe the characteristics of dependent variable related to independent. The results of these models are compared by means of cross validation and coefficient of determination. It is observed that generalized linear model perform poorly then generalized additive model and piecewise regression model. Since generalized additive model is non-parametric and is more flexible to model non-linear behavior therefore it also explains more variation of dependent variable then piecewise regression models. However,the difference between generalized additive model and piecewise regression model becomes negligible if segmented points are observed from generalized additive model and then those segmented points are used to estimate piecewise regression.

1. Introduction

In the recent decades, the whole world is witnessing and facing frequently the weather extreme events due to global warming. Some of these weather extreme events are potentially destructive in their nature. The major extreme events are cold waves, heat waves, cyclones, fog, thunderstorms, hailstorms, snowstorms, flash flood and unpredictable pattern of precipitations(Ali & Iqbal, 2012). Climate change, variability and inter annual fluctuations profoundly and strongly influence the social and natural environments throughout the whole world with consequent impacts and vulnerabilities on natural resources and industry (Adnan, 2009).

Pakistan is a country which is situated in the south Asian region between 61-76^{o}E longitude and 24-37^{o}N latitude near above the tropic of cancer. It has more diversity in its climate due to its altitude and topology. The local topology of the Pakistan can be described as Hindu Kush, Indus plain, Baluchistan plateau and Himalayas range (M. S. Hussain & L. Lee, 2009). The whole of Sind, southern Punjab, a major part of Baluchistan and some central parts of the Northern areas have arid or semi-arid. South west region of Pakistan is considered semi-arid to arid on the contrary to south east region. Generally 50% of Pakistan is arid, 40% is semi-arid and remaining 10% receive humidity (Khan, 2002).

The annual rainfall of Pakistan is low and very irregular, with national average of almost 278 mm (millimeter), varying from 160 mm during a dry year of 2002 to around 440 mm in a wet year of 1994. So this rainfall is contributing an average of 180 MAF (million acre-feet) of rainwater over the Pakistan total land area. The spatial variation or structure of rainfall is very huge and enormous in the entire country; as the northern mountainous area receives 1500 mm rainfall, northern Punjab regions receives less than 100 mm rainfall, southern Punjab and upper Sindh areas are getting 150 mm of rainfall per annum.

As the Pakistan is located at the western edge of monsoonal system so the heavy summer precipitins or rains occurs only in northern regions of the Punjab province while the rest of the regions of the country are in permanents state of water shortfall or shortage. The monsoon season in the Punjab province is highly varying from year to year. It is also expected that climate changes will enhance the variability of winter rains and monsoon seasons regarding shift in time and amounts.(Hussain & Lee, 2009).

In modeling the data, there may be a variety of relationships among dependent and independent variables but the linear functions and polynomial functions with abrupt changes are most common. Particularly in environment, medical, agriculture and in most of the phenomena of daily life, non-linear behavior is mostly observed. It is, certainly, due to the reason that the independent variables obtain a threshold level. Due to this turning point, the estimation via linear regression may not stand viable. So, one has to use non-linear methods, and piecewise regression is one of them.

Segmented regression was used by (Lerman 1980) in comparison with standard methods of linear regression and found it as an alternate tool for reliable inferences. He introduced the grid search methodology when there were more than one break point (Hockey stick) the multiple functions depending upon the value of the regressand were used earlier the (Lerman 1980). With standard notations of regression methodology the functional form at proceeding breakpoint seemed to like this $f_i(X_i, \beta_i) = f_{i+1}(X_i, \beta_{i+1})$ subject to this condition, the minimum value of residual sum of squares $\hat{S}$ stand to be non-trivial as $\hat{S}$ possesses potentially many local minima, and is non-differentiable at a number of points (specifically if any Xi coincides with the abscissa of an observation) which implies that some minima of $\hat{S}$ need not be stationary points.

Parameter estimates in such case sometimes lead to fallacious predictions. Also the initial value of the break points used to play the pivotal role earlier to (Lerman 1980). (Piepho and Ogutu 2003) used longitudinal data from ecology and pharmacology and compared the results of two data sets by using segmented regression technique.(Sprent 1961)used two - phase regression while testing the hypothesis about consistency of parameters, he used break-points but in the sense of two-phase regression rather than the segmented regression. However, (Sprent 1961) provided an idea of the later.(Kirch and Steinebach 2006) found approximate values for change point tests only without going towards the segmented or broken stick regression. They studied both the abrupt and gradual changes by using permutation principles. Segmented regression modeling with two partitioning variables can well be done by grid search method using SAS PROC NLIN tool, but this problem was tackled by (Kim and Kim 2008)quite differently, focusing on the asymptotic behavior of least square estimates rather than fitting algorithms to provide the asymptotic inference on the parameters. Extensions of the simple models with one change–point to the multiple change-points were also established and asymptotic inferences were made accordingly. Hence the estimated regression coefficients were proved to follow a multivariate normal distribution. The estimated change points (continuous case) with multiple normal distributions were proved to be consistent at $1/p_n$-rate. Also when the segments were considered not to be continuous at change points, the change points were proved to be consistent at the 1/n-rate.

(Chiu, Lockhart et al. 2006) tried to tackle irregular patters in non-linear models by defining few regularity conditions which suffice the difficulty caused by non-differentiability of the model's first partial derivatives. (Ansari, Gray et al. 2003) used the segmented regression to evaluate the proper use of alert antibiotics in a territory hospital of U.K. when a separate control group was not available. Breakpoint used by (Ansari, Gray et al. 2003), was the time at which the alert antibiotic policy was implemented on experimental basis in a hospital. Along with other relevant factors they found that the magnitude of the use of alert antibiotics reduced.

(Gillings, Makuc et al. 1981) while evaluating the mortality in connection with the historical trends in perinatal and post neonatal mortality and morbidity in North Carolina Regionalized Perinatal Care Program (RPC) used segmented regression as a tool. The yearly plot for the study region suggested a downward trend. Dropping the candidature of linear trend and a non-linear parabolic trend due to the reason that these models might be difficult to interpret and not appropriate for identifying shifts due to the "interruption" of regionalization, the authors used a piecewise regression. Perinatal and post neonatal mortality patterns were treated as segments in a interrupted time series design. With different mortality trends in blacks(including Indians) and whites and having variant time dependent mortality with the assumption of independent errors, (Gillings, Makuc et al. 1981) were at the point that segmented regression was a suitable tool for those types of analysis.

(Reynolds and Chiu 2010) in this regard state “Specifically, when an abrupt breakpoint estimate is forced on a relatively smooth or gradual profile, the range of time steps required to bracket that estimate will be necessarily much smaller than those required for a smooth curve. In these cases, the broken stick fit could result in estimates of CTP and confidence intervals that poorly represent the observed profile”

Regression Models which have different analytical forms in different regions of the domain of the independent variable have been studied by (Feder 1975) derived asymptotic distribution as chi-square under some suitable identifiable conditions of the log likelihood ratio statistic in said models. (Feder 1975) showed by example that if there were actually fewer segments than the number assumed in the model, then the least squares estimates were not asymptotically normal and the distribution of log likelihood ratio statistic did not stood asymptotically chi-square. He writes,” The asymptotic behavior is then more complicated, and depends on the configuration of the observation points of the independent variable.”

(Clegg, Hankey et al. 2009)used segmented regression while analyzing the trend using the annual average percent change in rat population and denoted it as (sAPCs). (Clegg, Hankey et al. 2009) assumed that the change in the population of rats was constant within each time partition but varied between (among) different time partitions. Also different groups (racial subgroups) might have different transition points and thus different constant rates in different partitions at a common time interval (past 10 years) were assumed to be held. By using sAPCs, the researchers proposed the average annual per cent change (AAPC) take into account the trend transitions. Contrasting the conventional annual per cent change (cAPC), working under the assumption of a constant rate of change, the new AAPC used the trend transitions. When there were no changes in trends for different partitions, the AAPC had the tendency to be reduced to the cAPC and sAPC and also the AAPC for any sub-time intervals of specified length stood exactly the same as the AAPC (and the sAPC) over the entire time interval, but the conventional cAPCproducd varying results for different sub-time intervals.

(Kim, Yu et al. 2008; Kim, Yu et al. 2009) by using join point software which analyze segmented regression while using Lerman’s grid search method (LGS) and Hudson’s continuous fitting algorithm (HCA) studied practical issues in constructing the confidence intervals for the parameters of segmented line regression. (Kim, Yu et al. 2009) indicated that these inferential procedures were reasonably robust for Poisson data. Kim et al also compared these two methods that is Lerman’s and Hudson’s and found thatHudson’s continuous fitting algorithm (HCA) performed better as compared to Lerman’s grid search method (LGS) while using simulations.

(Kim, Yu et al. 2009) estimated the change points and studied their asymptotic properties, in segmented regression by using his permutation technique. Kim compared the asymptotic efficiency of permutation method with the methods used to select the parsimonious models like AIC and BIC and found it at least equally efficient as that of BIC. Kim’s permutation procedure estimated consistent change points, under some constraints via simulations. Kim’s problem was only to study the asymptotic properties of estimated change points; it was different from the classical regression procedure for the selection of model which was quite evident due to the reason that change points involved a regression matrix with some unknown parameters.

(Barrowman and Myers 2000) suggested a piecewise linear spawner-recruitment model regarding the fisheries .This model perhaps is dependent on particular pattern, might be any of the exponential family being generated from the origin, and then it abruptly follows the straight line resembling the uniform distribution being parallel to x-axis. With a wide range of applications to different data sets, they were of the view that hockey stick model performed in relatively better way due to the R^2 and confidence intervals of their estimates.

In the present study we will consider two data sets to strengthen our findings. The first data we considered is a well-known ozone data set in which independent variables are behaving non-linearly. Second data set contains rainfall, maximum humidity and maximum temperature during monsoon period for two monitoring stations (Islamabad and Barkhan). Generalized linear models, Generalized additive model and piecewise regression models are used to find appropriate model to describe the characteristics of dependent variable related independent variables. The results of these models are compared by means of cross validation and coefficient of determination. It is observed that generalized linear model perform poorly then generalized additive model and piecewise regression model. Since generalized additive model is non-parametric and is more flexible to model non-linear behavior therefore it also explains more variation of dependent variable then piecewise regression models.

2. Material and Methods

2.1 Study Area

In the present study the precipitation data set with maximum temperature and maximum humidity as independent variable is used. The precipitation data is collected for 27 years period during monsoon from two monitoring stations (Islamabad and Barkhan) in Pakistan.

2.2 Generalized Linear Model

(Nelder and Wedderburn 1972) proposed Generalized Linear Model (GLM) which is an extension of the linear regression model. It allows the data to follow a certain probability distribution such as the Poisson, Binomial, Multinomial and normal distribution. In case data follow normal distribution then it becomes special case of regression model. Additionally, Generalized Linear Model relaxes the assumption of equality or constancy of variances which is the basic requirement for hypothesis tests in traditional linear models. In GLM response variable should belong to exponential family and link describes how the mean of response variable relates to linear combinations of predictors. In GLM, the distribution of response variable belongs to exponential family of distribution which could be written as

$$f(y|\theta,\varphi) = \left[\frac{y\theta - b(\theta)}{a(\varphi)} + c(y,\varphi)\right]$$

Where θ is called canonical parameter and represents the location and φ is scale parameter representing dispersion. For different values of $a, b$and c the various members of family can be specified.

2.2.1 Link Function

In generalized linear model, the effect of the predictors on their response through a linear predictor is modeled by using link functions i.e.

$$\eta = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p = X^T\beta$$

The link functionηdescribes how the mean response is linked to the covariates through the linear predictori.e.$\eta = g(\mu)$. Simply speaking any monotone continuous and differentiable can be a link function in GLM. In Gaussian linear model link function is identity i.e. $\eta = \mu$. In case of Poisson GLM the mean must be positive therefore in that case$\eta = \mu$ will not be appropiriate because η can be negative. The reasonable choice could be $\mu = e^{\eta}$ so that $\eta = \log \mu$ which will ensure $\mu > 0$, such *log* link will describe that additive effect of predictors leads to multiplication effect on μ. In case of binomial GLM, it is assumed that assumed that p is the probability of success and it also represents our μ and response is defined as proportion rather than count. Different link functions (logistic, probit and complementary log-log link) can be defined for binomial GLM the selection depends on the objective of the study. The link functions and variance functions for exponential family of distributions are shown in Table 1.

Table 1: Link Functions and Variance Functions for Exponential Family of Distributions

Family	Link	Variance Function
Normal	$\eta = \mu$	1
Poisson	$\eta = \log \mu$	μ
Binomial	$\eta = \log\left(\frac{\mu}{1-\mu}\right)$	$\mu(1-\mu)$
Gamma	$\eta = \frac{1}{\mu}$	μ^2
Inverse- Gamma	$\eta = \frac{1}{\mu^2}$	μ^3

2.3 Generalized Additive Model

Many nonparametric methods do not perform well when there is a large number of independent variable in the model. The sparseness of data in this setting inflates the variance of the estimates. The problem of rapidly increasing variance for increasing dimensionality is sometimes referred to as the "curse of dimensionality". Interpretability is another problem with nonparametric regression based on kernel and smoothing spline estimates (Hastie and Tibshirani 1990).

To overcome these difficulties, (Stone 1985) proposed additive models. These models estimate an additive approximation to the multivariate regression function. The benefits of an additive approximation are at least twofold. First, since each of the individual additive terms is estimated using a univariate smoother, the curse of dimensionality is avoided, at the cost of not being able to approximate universally. Second, estimates of the individual terms explain how the dependent variable changes with the corresponding independent variables.

(Hastie and Tibshirani 1990) proposed generalized additive models. These models assume that the mean of the dependent variable depends on an additive predictor through a nonlinear link function. Generalized additive models permit the response probability distribution to be any member of the exponential family of distributions. Many widely used statistical models belong to this general class, including additive models for Gaussian data, nonparametric logistic models for binary data, and nonparametric log-linear models for Poisson data.

Suppose that Y is a response random variable and $X_1, \ldots, X_p$ are the predictors. The ordinary least square linear regression model assumes that expected value of Y has a linear relationship with response variable as;

$$E(Y) = f(X_1, \ldots\ldots, X_p) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

Where the parameters $\beta_0, \beta_1, \ldots\ldots \beta_p$ are estimated by method of least square.

The generalized additive models works in similar fashion as linear regression or generalized linear regression model performs. The difference is only generalized additive model use non-parametric smoothing functions to describe the relationship of response variable and predictors whereas linear regression or generalized linear regression models use parametric functions to estimate relationship of response variable and predictors. Moreover, generalized additive models are more flexible and can account the non-linear behavior of predictors.
In generalized additive model the expected value of response variable related to set of predictors $X_1, \ldots X_p$ is modeled by using smooth functions;

$$E(Y) = f(X_1, \ldots\ldots, X_p) = \beta_0 + f_1(x_1) + \cdots + f_p(x_p) = \beta_0 + \sum_{j=1}^{p} f_j(x_j)$$

Wheref_j, $j = 1, \ldots, p$ are smooth functions and are estimated in a nonparametric fashion.

2.4 Piecewise Regression Models

Ryan and Porth (2007) described when our objective stand to analyze a relationship between a dependent variable and a set of predictors X, there may be possibility that for different ranges of X the different linear relationships exists. In such situations using a single linear model will be inappropriate and non-linear parametric model may not exist. Piecewise linear regression is a type of regression which allows multiple linear models to be fitted to the data for different ranges of x. Breakpoints are defined for different ranges of predictors when the slope of linear function changes. The value of the breakpoint may or may not be known before the analysis, but typically it is unknown and must be estimated. The regression function at the breakpoint may be discontinuous, but a model can be written in such a way that the function is continuous at all points including the breakpoints. When there is only one breakpoint, at x=c, the model can be written as follows:

$$y = a_1 + b_1 x for x \leq c$$
$$y = a_2 + b_2 x for x > c$$

Maintaining the regression function to be continuous at the breakpoint, the two equations for $y$need to be equal at the breakpoint (when$x = c$) i.e.$a_1 + b_1 c = a_2 + b_2 c$. The solution for parameters in terms of others parameters can be obtained as

$a2 = a1 + c(b1 - b2)$.

Then by replacing $a2$ with the equation above, the result is a piecewise regression model that is continuous at $x = c$:

$y = a1 + b1x$ *for* $x \leq c$
$y = \{a1 + c(b1 - b2)\} + b2x$ *for* $x > c$.

The non-linear least square techniques can be used to estimate the parameter of piecewise regression. For estimation of parameters in piecewise regression we utilized segment package by Muggeo (2008).

3. Results and Discussions

Scatter diagram are drawn to observe the relationship of variables with each other. In Figure **1** row 3 it could be observed that Box-Cox transformed rainfall data is non-linearly related with maximum humidity and minimum temperature.

In **Table 1** the summary statistics of generalized linear model is presented. In last column the p-values are provided which show that all independent variables have significant effect on the dependent variable. The coefficient of determination (R^2) shows that 60.63% percent of variation of dependent variable is explained by independent variables.

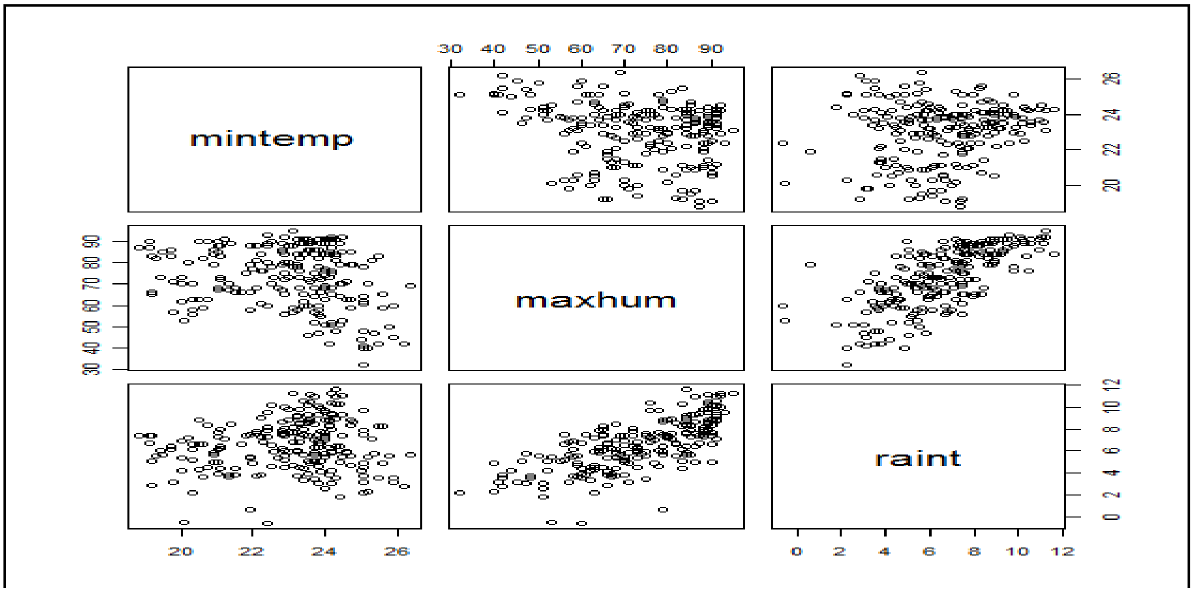

Figure 1: Scatter Plot of Precipitation Data Set.

Table 1: The Estimated Parameters using Generalized Linear Regression Model

Coefficients	Estimate	Standard Error	t-values	Pr(>|t|)
Intercept	-14.031	1.600	-8.768	5.85e-16 ***
Minimum Temperature	0.47	0.06	-5.105	1.45e-06 ***
Maximum Humidity	0.13	0.01	18.44	2e-16 ***

Significant. Codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

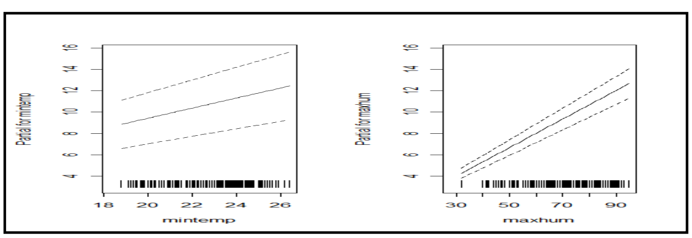

Figure 2: The Estimated Partial Effect of Minimum Temperature and Maximum Humidity on Precipitation by Using Generalized Additive Model

Since the independent variables are non-linearly related to dependent variable therefore we used generalized additive model. The partial effects of minimum temperature and maximum humidity on the rainfall data is displayed in **Figure 2**.It could be observed from **Figure 2** that generalized additive model looks inappropriate fit in present situation. It represents that independent variables are linearly behaving with dependent variable whereas in actual situation maximum humidity is behaving non-linearly. The value of estimated adjusted R^2 is 0.62 which means the independent variables explained 62% variation of dependent variable whereas 38% variation of dependent variable remains unexplained.

Parametric coefficients:

Table 2: The Estimated Parameter Based on Generalized Additive Model

	Estimate	Standard error	t-values	p-value
Intercept	-14.03	1.60	-8.77	5.85e-16 ***
Approximate significance of smooth terms				
Predictor	Edf	Ref.df	F-Statistic	p-value
s(Minimum Temperature)	0.47	0.06	7.85	2.01e-13 ***
s(maximum Humidity)	0.13	0.01	18.44	2e-16 ***

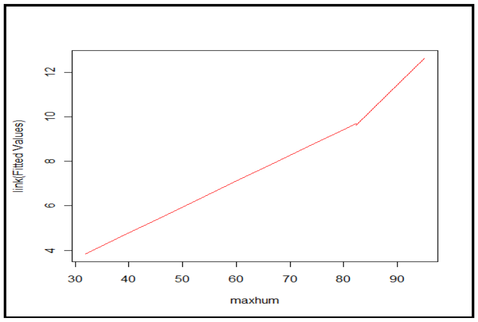

Figure 3: The Partial Effect of Maximum Humidity on Precipitation Estimated by Piecewise Regression Model

In **Table 3** the summary statistics of piecewise regression are provided. The coefficients of predictors in column 2 are estimated in two steps. In first step generalized linear model is fitted with Gaussian link function and in next step piecewise regression model is fitted. It can be observed that all predictors have significant effect on the transformed rainfall data set. The overall estimated piecewise regression model is also significant. The estimated breakpoint of maximum humidity is 82.240 with standard error 3.364. The variable minimum temperature is nearly related with dependent variable and has significant effect on dependent variable. The value of adjusted R^2 for fitted piecewise regression model is 0.63 which shows that 63% variation of transformed rainfall data set is explained by independent variables and 37% of dependent variable remains unexplained. The partial relationship of maximum humidity on transformed rainfall data set is shown in Figure 3.

Table 3: Estimated Parameters by using Piecewise Regression Model

Coefficients	Estimate	Standard Error	t-values	Pr(>|t|)
Intercept	-12.56	1.73	-7.28	6.58e-12 ***
Minimum Temperature	0.45	0.06	7.56	1.25e-12 ***
Maximum Humidity	0.12	0.01	10.46	2e-16 ***
U1.Maximum Humidity	0.12	0.06	2.12	NA

Signif.codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

3.2 Conclusion

In modeling data, there may be a variety of relationships among dependent and independent variables but the linear functions and polynomial functions with abrupt changes are most common. Particularly in environment, medical, agriculture and in most of the phenomena of daily life, non-linear behavior is mostly observed. It is, certainly, due to the reason that the independent variables obtain a threshold level. Due to this turning point, the estimation via linear regression may not stand viable. So, one has to use non-linear methods, and piecewise regression is one of them.

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It is also expected that climate changes will enhance the variability of winter rains and monsoon seasons regarding shift in time and amounts.(Hussain & Lee, 2009)

For the validation of appropriate method the precipitation data set with maximum temperature and maximum humidity as independent variable is used. The precipitation data is collected for 27 years period during monsoon from two monitoring stations (Islamabad and Barkhan) in Pakistan.

In the present study generalized linear model, generalized additive model and piecewise regression models are compared. It is concluded that with generalized additive model the independent variables are explaining higher percentage of variation for dependent variable which is comparatively higher then generalized linear model however in segmented regression model more variation is explained by independent variables for dependent variable. The drawback of generalized additive model is that it is non-parametric and we could not estimate the values of parameters like in generalized linear model or piecewise regression model. However the use GAM model can provide us reliable information about the relationship of dependent and independent variables which could provide reasonable direction to fit piecewise regression model. It is suggested that at first step, we should fit GAM model and observe the behavior of dependent and independent variables. In second step, we should use piecewise regression model in a result it will explain more percentage of variation of dependent variable; additionally we have benefit of having parametric model.

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