An Efficiency Comparison of Certain Randomized Response Strategies

Muhammad Abid
National Accounts Section, Federal Bureau of Statistics, Islamabad, Pakistan.

Salman Arif Cheema
Department of Statistics, Virginia Tech (VA), Blacksburg 26040, USA.

Zawar Hussain*
Quaid-i-Azam University, Islamabad, Pakistan.
E-mail: zhlangah@yahoo.com*

Abstract

Several researchers have already performed a comparison between different randomized response strategies, but have concentrated solely on comparing the variances of the appropriate estimators. A very little attention has been paid by these researchers to the degree of privacy protection offered to the interviewees. In this paper, an attempt has been made in this direction. Some important randomized response strategies such as Warner (1965), Mangat et al. (1995), Bhargava and Singh (2000) have been compared with Mahmood et al. (1998) model, taking into account the aspect of equal privacy protection. Based on the concept of jeopardy function, we derived efficiency conditions for the Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) model compared to Mahmood et al. (1998) model.

Key words: Randomized response technique; privacy protection; jeopardy function; efficiency comparison

1 Introduction

Randomized response procedure is used in sample surveys of human populations for estimating the proportion of a population possessing a given attribute. They are mostly appropriately used when the attribute under study is such that possessors of the attribute show a tendency to conceal having it when confronted with a interviewer's direct question. In a typical situation, the attribute may concern the respondent's involvement or non-involvement in illegal or socially deviant behavior. Since the degree of privacy is an essential part of the randomized response procedure and greater privacy will be attained in general, in terms of variances, only when the required degree of protection is held constant. To achieve this, we have considered a measure of privacy protection which was given by Leysieffer and Warner (1976). This measure of privacy protection has also been used by Bhargava and Singh (2002) and Tung-Hai Lin (2005).According to Leysieffer and Warner (1976), a population is divided into complementary sensitive groups, A and A^C with unknown proportion π and (1- π), respectively. Let us consider a dichotomous response model where a typical response *R* is *yes* (say, *y*) or *no* (say, *n*). The conditional probabilities that a researcher *R* comes from individual of groups, A and A^C, are $P(R|A)$ and $P(R|A^C)$, respectively. These probabilities are at the investigator's disposal and are called design probabilities. Using these design probabilities, Leysieffer and Warner (1976) proposed the natural measure of jeopardy carried by *R* about A and A^C, respectively. These measure are as follows:

$$g(R|A) = \frac{P(R|A)}{P(R|A^C)}; \qquad g(R|A^C) = \frac{1}{g(R|A)}. \tag{1.2}$$

In the sequel, we will call these measures as jeopardy functions.

They have also shown that an unbiased estimator for π is defined if and only if

$$P(y|A) - P(y|A^C) \neq 0, \tag{1.3}$$

and the existence of an unbiased estimator for π necessarily makes a response jeopardic with respect to either A and A^C.

Assuming, without loss of generality, that

$$P(y|A) > P(y|A^C), \tag{1.4}$$

so that a "yes" answer increases the odds of A and is jeopardizing with respect to A, i.e.

$$g(y|A) = \frac{P(y|A)}{P(y|A^C)} > 1, \tag{1.5}$$

while a "no" answer increases the odds of A^C and is jeopardizing with respect to A^C, i.e.

$$g(n|A^C) = \frac{P(n|A^C)}{P(n|A)} > 1. \tag{1.6}$$

Therefore for the sake of efficiency, one can needs as large magnitudes as possible for $g(y|A)$ and $g(n|A^C)$ and both above unity. Hence for the practical point of view, regarding protection of privacy one can fix some maximal allowable levels of $g(y|A)$ and $g(n|A^C)$ (say, k_1 and k_2), respectively. After fixing $g(y|A)$ and $g(n|A^C)$ at k_1 and k_2, the optimal choice of the design parameters for the particular randomized response model can be worked out. These design parameters will now be in terms of k_1 and k_2. In this way, we can derive the variance expression for each randomized response model by substituting the values of design parameters and then these variances can be at the same level of protection of privacy.

In the next section, we give the short introduction of the strategies proposed by Warner (1965), Mangat et al. (1995), Bhargava and Singh (2000) and Mahmood et al. (1998) randomized response techniques.

2 Mahmood et al. (1998) RR Model

Mahmood et al. (1998) provides three simple and alternative survey techniques which parallel the optimality conditions suggested Moors (1971) and are free from the difficulties in Moors (1971), as well as in Mangat et al. (1977) model. In this section, we use the technique-I which is better than the other two techniques, we are suggesting a device consisting of three types of statements: (i) "I belong to group *A*" (ii) "I belong to group Y^C" and (iii) "I belong to group *Y*", with probabilities P_1, P_3 and P_4 respectively, such that $P_1 + P_3 + P_4 = 1$. In the second independent sample of n_2 respondents, the question is asked only on the unrelated question "Y" to estimate the proportion π_Y of the unrelated character as discussed in Moors model. Obviously this technique guarantees the respondents privacy which is lacking in Moors model. The unbiased estimator of π, in this case, is

$$\hat{\pi}_{mm} = \frac{n'/n - p_3(1-\pi_y) - p_4\pi_y}{p_1}, \quad (2.1)$$

where n'/n is the observed proportion of "yes" answers obtained from the respondents selected by simple random sampling with replacement.

The variance of the estimator $\hat{\pi}_{mm}$ is given by

$$V(\hat{\pi}_{mm}) = \frac{\left[\sqrt{\hat{\lambda}_3(1-\hat{\lambda}_3)} + (p_3 - p_4)\sqrt{\pi_y(1-\pi_y)}\right]^2}{np_1^2}. \quad (2.2)$$

3 Warner (1965) RR Model

Warner (1965) proposed a randomized response technique, which design two types of cards, one type is "*I* belong to group *A*" and the other is "*I* do not belong to group *A*", represented with probabilities P_1 and $1-P_1$,respectively. Warner derived an unbiased estimator of π is given by

$$\hat{\pi}_w = \frac{n'/n - (1-p_1)}{2p_1 - 1}, \qquad p_1 \neq \frac{1}{2} \quad (3.1)$$

where n'/n is the observed proportion of "yes" answers obtained from the respondents selected by simple random sampling with replacement.

The variance of the estimator $\hat{\pi}_w$ is given by

$$V(\hat{\pi}_w) = \frac{\pi(1-\pi)}{n} + \frac{p_1(1-p_1)}{n(2p_1-1)^2}. \quad (3.2)$$

4 Mangat et al. (1995) RR Model

Mangat et al. (1995) presented a technique modified from warner's model. They design three types of cards, which are "*I* belong to group *A*", "*I* do not belong to group *A*" and left blank, represented with probabilities P_1, P_2 and P_3, respectively, $P_1 + P_2 + P_3 = 1$. If the blank card is drawn by the respondent, who must reply "no", no matter what status he/she has. Under this procedure an unbiased estimator of π is given by

$$\hat{\pi}_m = \frac{n'/n - p_2}{p_1 - p_2}, \qquad p_1 \neq p_2 \tag{4.1}$$

where n'/n is the observed proportion of "yes" answers obtained from the respondents selected by simple random sampling with replacement.

The variance of the estimator $\hat{\pi}_m$ is given by

$$V(\hat{\pi}_m) = \frac{\pi(1-\pi)}{n} + \frac{\pi p_3}{n(p_1 - p_2)} + \frac{p_2(1-p_2)}{n(p_1-p_2)^2} \tag{4.2}$$

5 Bhargava and Singh (2000) RR Model

Bhargava and Singh (2000) proposed a randomized response procedure just the same as Mangat et al. model only with a little difference which is that, when the blank card is drawn by the respondent , who is required to say "yes" only. Suppose these three types of cards are represented with probabilities P_1', P_2' and P_3' respectively, where $P_1' + P_2' + P_3' = 1$. Under this procedure an unbiased estimator of π is given by

$$\hat{\pi}_b = \frac{n'/n - (p_2' + p_3')}{p_1' - p_2'}, \qquad p_1' \neq p_2'. \tag{5.1}$$

where n'/n is the observed proportion of "yes" answers obtained from the respondents selected by simple random sampling with replacement.

The variance of the estimator $\hat{\pi}_b$ is given by

$$V(\hat{\pi}_b) = \frac{1-\pi}{n} - \frac{\pi p_3'}{n(p_1' - p_2')} + \frac{p_1'(1-p_1')}{n(p_1'-p_2')^2}. \tag{5.2}$$

In this section, based on the concept of jeopardy function, we derived efficiency conditions for the Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) model compared to Mahmood et al. (1998) model at equal level of privacy protection.

6 Efficiency Conditions for Mahmood et al. (1998) RR Model at Equal Level of Privacy Protection

Considering the Mahmood et al. (1998) model, we have the design probabilities as

$$P(y|A) = p_1 + p_3(1-\pi_y) + p_4\pi_y. \tag{6.1}$$

$$P(y|A^C) = p_3(1-\pi_y) + p_4\pi_y. \tag{6.2}$$

$$P(n|A) = p_4(1-\pi_y) + p_3\pi_y. \tag{6.3}$$

$$P(n|A^C) = p_1 + p_4(1-\pi_y) + p_3\pi_y. \tag{6.4}$$

From (1.2) and (6.1-6.4), using *MM* to index the jeopardy functions for the Mahmood et al. (1998) model, we have

$$g_{mm}(y|A) = \frac{P(y|A)}{P(y|A^C)} = \frac{p_1 + p_3(1-\pi_y) + p_4\pi_y}{p_3(1-\pi_y) + p_4\pi_y},$$

$$= 1 + \frac{p_1}{p_3(1-\pi_y) + p_4\pi_y}. \tag{6.5}$$

$$g_{mm}(n|A^C) = \frac{P(n|A^C)}{P(n|A)} = \frac{p_1 + p_4(1-\pi_y) + p_3\pi_y}{p_4(1-\pi_y) + p_3\pi_y},$$

$$= 1 + \frac{p_1}{p_4(1-\pi_y) + p_3\pi_y}. \tag{6.6}$$

as we know that P_1, P_3 and P_4 are the design probabilities and π_y estimate the proportion of unrelated question, so the terms $\frac{p_1}{p_3(1-\pi_y) + p_4\pi_y}$ and $\frac{p_1}{p_4(1-\pi_y) + p_3\pi_y}$ are always positive. So it is identifies that "yes" and "no" as jeopardy for A and A^C respectively.

Let k_1 and k_2 be the maximal allowable values of $g_{mm}(y|A)$ and $g_{mm}(n|A^C)$. If $k_1 = k_2 = k$, say, maximization of $g_{mm}(y|A)$ and $g_{mm}(n|A^C)$ leads to design with

$$1 + \frac{p_1}{p_3(1-\pi_y) + p_4\pi_y} = k_1 \quad (6.7)$$

and

$$1 + \frac{p_1}{p_4(1-\pi_y) + p_3\pi_y} = k_2. \quad (6.8)$$

Now solving (6.7) and (6.8) by taking $k_1 = k_2$, we get

$$p_1 = \frac{(k_1-1)\left[(1+(1-\pi_y))(2\pi_y-1)\right]}{(2\pi_y-1)\left[(1+(1-\pi_y)(k_1-1))\right] + k_1(2\pi_y-1)} \quad (6.9)$$

and

$$p_3 = p_4 = \frac{(2\pi_y-1)}{(2\pi_y-1)\left[(1+(1-\pi_y)(k_1-1))\right] + k_1(2\pi_y-1)}. \quad (6.10)$$

Hence, (6.9) and (6.10) is the optimal choice for design parameter p_1, p_3 and p_4 of the Munir's model, with this optimal choice of p_1, p_3 and p_4, the variance of the unbiased estimator $\hat{\pi}_{mm}$ is given by

$$V(\hat{\pi}_{mm}) = \frac{\pi(1-\pi)}{n} + \frac{(2\pi_y-1)\times a}{n\times b^2} \quad (6.11)$$

where

$$a = \left[(k_1-1)^2\pi - (k_1-1)(1-\pi_y)\left\{2\pi_y(k_1-1)\pi - (2\pi_y-1)\right\} + k_1(2\pi_y-1)\right] - (k_1-1)^2\pi$$

and $b = \left[(k_1-1)\left\{(1+(1-\pi_y))(2\pi_y-1)\right\}\right]$

In the same pattern, we can find efficiency conditions for Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) RR model at equal level of privacy protection. These efficiency conditions are given below in Table1.

Efficiency conditions for Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) RR model at equal level of privacy protection.

Model	Design Probabilities	Jeopardy Function	Condition for Jeopardy	Optimal Choices for Design Parameters	Optimal Variance
Warner (1965)	$P(y\|A) = P(n\|A^C) = p_1$ $P(n\|A^C) = P(y\|A^C) = 1-p_1$	$g_w(y\|A) = \frac{P(y\|A)}{P(y\|A^C)} = \frac{p_1}{1-p_1}$ $g_w(n\|A^C) = \frac{P(n\|A^C)}{P(n\|A)} = \frac{p_1}{1-p_1}$	$p_1 > 1/2$ assures that "yes" and "no" are jeopardy for A and A^C respectively.	$p_1 = \frac{k_1}{k_1+1}$	$V(\hat{\pi}_w) = \frac{\pi(1-\pi)}{n} + \frac{k_1(k_1-1)^{-2}}{n}$
Mangat (1998)	$P(y\|A) = P(n\|A^C) = p_1$ $P(n\|A^C) = P(y\|A^C) = p$	$g_m(y\|A) = \frac{P(y\|A)}{P(y\|A^C)} =$ $g_m(n\|A^C) = \frac{P(n\|A^C)}{P(n\|A)} = \frac{1-}{1-}$	$p_1 > p_2$ assures that "yes" and "no" are jeopardy for A and A^C respectively.	$p_1 = \frac{k_1(k_1-1)}{k_1^2-1}$ $p_2 = \frac{k_1-1}{k_1^2-1}$	$V(\hat{\pi}_m) = \frac{\pi(1-\pi)}{n} + \frac{(1-\pi)}{n(k_1-1)}$
Bhargava and Singh (2002)	$P(y\|A) = P(n\|A^C) = p_1' + p$ $P(n\|A^C) = P(y\|A^C) = p_2' + p$	$g_b(y\|A) = \frac{P(y\|A)}{P(y\|A^C)} = \frac{1-}{1-}$ $g_b(n\|A^C) = \frac{P(n\|A^C)}{P(n\|A)} =$	$p_1' > p_2'$ assures that "yes" and "no" are jeopardy for A and A^C respectively.	$p_1' = \frac{k_1(k_1-1)}{k_1^2-1}$ $p_2' = \frac{k_1-1}{k_1^2-1}$	$V(\hat{\pi}_b) = \frac{1-\pi}{n} + \frac{k_1}{n}$

7. Comparisons

Here, we computed the percent relative efficiencies of Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) relative to Mahmood et al. (1998) model as

$$PRE = \frac{Var(\hat{\pi}_i)}{Var(\hat{\pi}_{mm})} \times 100, \qquad \text{where } i = w,\ m,\ b. \qquad (7.1)$$

The percent relative efficiencies (*PREs)* of the estimators $\hat{\pi}_w$, $\hat{\pi}_m$ and $\hat{\pi}_b$ with respect to $\hat{\pi}_{mm}$ are shown in Tables 2-7 (see Appendix), for the practicable choices of k_1, n and for the whole range of π_Y and π. These magnitudes help the investigators to choose the best strategy in different situations.

8 Conclusions

We can observe from Tables 2-7, that at equal level of protection the Mahmood et al. (1998) model is more efficient than the Warner (1965) model for the whole range of π_Y and π when k_1 =1.1 and n =30-50. It has been observed that the Mahmood et al. (1998) model performs better than the Warner (1965) model for the whole range of π, when k_1 =1.1, 1.3 and π_y =0.3, 0.7, for small as well as moderate sample sizes. It has also been observed that the Mangat et al. (1995), Bhargava and Singh (2000) models are more efficient than the Mahmood et al. (1998) model for the whole range of π_y and π, when we take k_1 =1.3, 1.5, 1.7 and 1.9 for small as well as moderate sample sizes. We can conclude from our study that the Mahmood et al. (1998) model can only perform better than the Warner (1965) model when the level of privacy protection is very small. It is also observed that as the sample size increases the efficiency of the Mahmood et al. (1998) model decreases as compared to the Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) models.

So, it may be concluded that the Mahmood et al. (1998) model cannot perform better than the Warner (1965), Mangat et al. (1995) and Bhargava and Singh (2000) models when the level of privacy protection is not very small for small as well as moderate sample sizes. From Tables 2-7, it is also observed that the Mangat et al. (1995) model is more efficient among the four models, when we study them at equal level of protection. Mangat et al. (1995) model also performs well among the four models, when the level of privacy protection is high for small as well as large sample sizes. So, it may be concluded that when the interviewee requires a high level of privacy protection, we can apply Mangat et al. (1995) model to collect data from the respondents in a small as well as large sample sizes.

References

1. F.W. Leysieffer and S.L. Warner, Respondent jeopardy and optimal design in randomized response models, Journal of the American Statistical Association, 171 (1976), 649-656.
2. J. J. A. Moors, Optimizing of the unrelated question randomized response model, Journal of the American Statistical Association, 66 (1971), 627-629.
3. M. Bhargava and R. Singh, A modified randomization device for Warner model, Statistica, 60 (2000), 315-321.
4. M. Bhargava and R. Singh, On the efficiency of the certain randomized response strategies, Metrika, 55 (2002), 191-197.
5. M. Mahmood, S. Singh and S. Horn, On the confidentiality guaranteed under randomized response sampling: a comparison with several new techniques, Biometrical Journal, 40 (1998), 237-242.
6. N. S. Mangat, R. Singh and S. Singh, Violation of respondent's privacy in Moors' model- Its rectification through a random group strategy, Communication in Statistics- Theory and Method, 26 (1997), 743-754.
7. N. S. Mangat, S. Singh and R. Singh, On the use of a modified randomization device in Warner model, Journal of Indian Society of Statistics and Operations Research, 16 (1995), 65-69.
8. S. L. Warner, Randomized response: a survey technique for eliminating evasive answer bias, Journal of the American Statistical Association, 60 (1965), 63-69.
9. Tung-Hai. Lin, The efficiency comparison on Warner's and two modified models, Journal of Information and Management Sciences, 16 (2005), 73-78.

Appendix

Table 2: *PRE of the estimator* $\hat{\pi}_{mm}$ *with respect to* $\hat{\pi}_w$ *for the whole range of* π *and* π_y, *when* k_1 =1.1, 1.3, 1.5, 1.7 *and* 1.9, *and* n =30.

$\pi_y = 0.1$					
	k_1				
π	1.1	1.3	1.5	1.7	1.9
0.1	481.267	48.4262	16.46181	8.123044	4.824594
0.3	481.9164	49.35354	17.22887	8.776497	5.399712
0.5	483.9095	50.16541	17.85021	9.310576	5.886536
0.7	487.2827	50.86374	18.31695	9.709996	6.26692
0.9	492.1029	51.44867	18.61676	9.95422	6.514995
$\pi_y = 0.3$					
0.1	2007.924	205.8967	70.9812	35.43549	21.26492
0.3	2137.372	221.3231	78.4169	40.65581	25.52124
0.5	2321.109	240.755	87.0633	46.63525	30.55938
0.7	2585.002	266.1696	97.66978	53.94301	37.07193
0.9	2977.825	301.0527	111.5868	63.71668	46.71606
$\pi_y = 0.7$					
0.1	2137.198	225.1720	77.93527	38.56652	22.76349
0.3	2424.642	245.8613	82.33309	39.59885	22.80266
0.5	2856.449	272.8222	86.91556	40.10516	22.30791
0.7	3562.099	309.7063	91.8742	40.09566	21.32654
0.9	4896.724	363.6013	97.47747	39.55333	19.88076
$\pi_y = 0.9$					
0.1	517.0354	56.61156	20.15697	10.19613	6.125499
0.3	520.0144	56.6028	19.99446	10.0257	5.969421
0.5	524.6318	56.46077	19.63906	9.6834	5.669558
0.7	530.9832	56.18623	19.10177	9.189208	5.250751
0.9	539.2094	55.77752	18.39001	8.558136	4.731079
$\pi_y = 1$					
0.1	333.404	37.021	13.31439	6.784956	4.099090
0.3	333.9697	36.89437	13.16794	6.651099	3.980808
0.5	335.1064	36.6445	12.88660	6.4017	3.767314
0.7	336.8276	36.27338	12.47990	6.051625	3.476168
0.9	339.1559	35.7815	11.95524	5.612406	3.120265
$\pi_y = 0$					
0.1	305.5085	30.40997	10.26722	5.041983	2.983621
0.3	305.7567	30.96555	10.72817	5.432835	3.325758
0.5	306.4628	31.41517	11.08647	5.742011	3.606362
0.7	307.6343	31.75738	11.33522	5.959005	3.813024
0.9	309.285	31.98953	11.46569	6.070415	3.928831

Table 3: *PRE of the estimator* $\hat{\pi}_{mm}$ *with respect to* $\hat{\pi}_w$ *for the whole range of* π *and* π_y*, when* k_1=1.1, 1.3, 1.5, 1.7 *and* 1.9, *and* n=50.

$\pi_y = 0.1$					
	k_1				
π	1.1	1.3	1.5	1.7	1.9
0.1	288.9963	29.07382	9.882078	4.875906	2.895829
0.3	289.5558	29.64413	10.34652	5.269928	3.242023
0.5	290.517	30.11306	10.71424	5.588225	3.532998
0.7	291.8896	30.4789	10.97813	5.820285	3.756726
0.9	293.6888	30.73883	11.12929	5.952586	3.896509
$\pi_y = 0.3$					
0.1	119.3758	122.6604	42.33948	21.15493	12.70294
0.3	124.0208	129.3647	46.02997	23.92940	15.04898
0.5	130.1548	136.9896	49.92537	26.86121	17.64485
0.7	138.1951	145.8827	54.15937	30.06684	20.67876
0.9	148.7857	156.5531	58.93377	33.74456	24.53533
$\pi_y = 0.7$					
0.1	125.5048	132.8004	46.10082	22.86264	13.51670
0.3	134.8080	139.1192	47.15071	22.87361	13.25606
0.5	147.0218	146.3169	47.88946	22.51052	12.68729
0.7	163.4571	154.7126	48.32218	21.79727	11.85466
0.9	186.4037	164.7672	48.44253	20.74574	10.78784
$\pi_y = 0.9$					
0.1	309.9933	33.94624	12.08792	6.114939	3.673853
0.3	310.9675	33.86885	11.96933	6.003811	3.575726
0.5	312.4262	33.6717	11.72512	5.786198	3.390016
0.7	314.3856	33.35675	11.36404	5.475567	3.132586
0.9	316.8695	32.92462	10.89293	5.082472	2.815195
$\pi_y = 1$					
0.1	200.0182	22.21034	7.987933	4.070663	2.459288
0.3	200.1635	22.11658	7.894737	3.988064	2.387146
0.5	200.4545	21.93201	7.71605	3.83442	2.257112
0.7	200.8932	21.65895	7.458564	3.619307	2.080155
0.9	201.4824	21.29901	7.127809	3.350327	1.864421
$\pi_y = 0$					
0.1	183.4375	18.25591	6.163032	3.026304	1.790743
0.3	183.7293	18.60055	6.442911	3.262302	1.996858
0.5	184.1336	18.86927	6.65779	3.447868	2.165321
0.7	184.6528	19.06021	6.80287	3.576214	2.288292
0.9	185.2899	19.17094	6.872426	3.638898	2.355247

Table 4: *PRE of the estimator $\hat{\pi}_{mm}$ with respect to $\hat{\pi}_m$ for the whole range of π and π_y, when k_1=1.1, 1.3, 1.5, 1.7 and 1.9, and n=30.*

$\pi_y = 0.1$					
	k_1				
π	1.1	1.3	1.5	1.7	1.9
0.1	39.73764	10.29533	5.108838	3.139581	2.159073
0.3	31.52724	8.56549	4.466744	2.886231	2.087005
0.5	23.04331	6.543314	3.570043	2.413853	1.826856
0.7	14.1927	4.199758	2.389168	1.685206	1.332337
0.9	4.872306	1.498505	0.8865124	0.6512106	0.5379354
$\pi_y = 0.3$					
0.1	165.7919	43.7733	22.02865	13.69593	9.516345
0.3	139.8281	38.41144	20..3031	13.370030	9.864036
0.5	110.5290	31.40282	17.41266	12.09062	9.483947
0.7	75.29133	21.97731	12.73954	9.36201	7.881433
0.9	29.48342	8.768526	5.313659	4.168381	3.85729
$\pi_y = 0.7$					
0.1	176.4659	47.87122	24.18681	14.90608	10.18698
0.3	1.58.6215	42.67014	21.34562	13.02244	8.813297
0.5	136.0214	35.58551	17.38311	10.39763	6.923145
0.7	103.7504	25.57208	11.98359	6.958752	4.533988
0.9	48.48242	10.59033	4.641784	2.587601	1.641531
$\pi_y = 0.9$					
0.1	42.69099	12.03553	6.255613	3.940837	2.741245
0.3	34.01963	9.823627	5.183749	3.297042	2.307200
0.5	24.98247	7.364448	3.927811	2.510511	1.759518
0.7	15.46553	4.63923	2.491535	1.594821	1.116301
0.9	5.338707	1.624588	0.8757147	0.559878	0.3906395
$\pi_y = 1$					
0.1	27.52877	7.870606	4.132051	2.622406	1.834399
0.3	21.84848	6.403155	3.41391	2.187274	1.538595
0.5	15.95745	4.779717	2.577320	1.6597	1.169166
0.7	9.810513	2.995050	1.627814	1.050282	0.739028
0.9	3.357979	1.042180	0.5692972	0.3671667	0.2576366
$\pi_y = 0$					
0.1	25.22548	6.465111	3.186378	1.948742	1.335212
0.3	20.00277	5.374186	2.781377	1.786637	1.285416
0.5	14.59347	4.097631	2.217295	1.488670	1.119216
0.7	8.960223	2.622169	1.478507	1.034208	0.8106428
0.9	3.062228	0.931734	0.5459851	0.3971299	0.3243989

Table 5: *PRE of the estimator* $\hat{\pi}_{mm}$ *with respect to* $\hat{\pi}_m$ *for the whole range of* π *and* π_y, *when* k_1 =1.1, 1.3, 1.5, 1.7 *and* 1.9, *and* n =50.

$\pi_y = 0.1$					
	k_1				
π	1.1	1.3	1.5	1.7	1.9
0.1	23.86208	6.181049	3.066852	1.884553	1.295923
0.3	18.9429	5.14485	2.682432	1.733064	1.253052
0.5	13.83414	3.92779	2.142848	1.448799	1.096448
0.7	8.50164	2.516606	1.431930	1.010132	0.798674
0.9	2.90781	2.90781	0.5299663	0.3894215	0.3217301
$\pi_y = 0.3$					
0.1	98.56718	26.07741	13.13984	8.176444	5.684741
0.3	81.13512	22.45172	11.93370	7.869401	5.816475
0.5	61.97846	17.86821	9.985074	6.964019	5.475989
0.7	40.25099	12.04536	7.064266	5.218213	4.396271
0.9	14.73126	4.5598	2.80637	2.207588	2.025853
$\pi_y = 0.7$					
0.1	103.6278	28.23316	14.30715	8.83648	6.048909
0.3	88.19213	24.14465	12.22426	7.522194	5.123506
0.5	70.01038	19.08482	9.577893	5.83606	3.937436
0.7	47.60886	12.77444	6.302892	3.782997	2.520281
0.9	18.45582	4.799046	2.306787	1.357198	0.8907394
$\pi_y = 0.9$					
0.1	25.59578	7.216917	3.751424	2.363443	1.6441
0.3	20.34367	5.878066	3.103160	1.974408	1.382029
0.5	14.87744	4.391961	2.345025	1.500125	1.052074
0.7	9.156863	2.754227	1.482266	0.950305	0.6659829
0.9	3.137322	0.9589695	0.5187111	0.3324982	0.2324473
$\pi_y = 1$					
0.1	16.51526	4.721883	2.479014	1.573324	1.100565
0.3	13.09481	3.838414	2.046784	1.311511	0.9226394
0.5	9.545455	2.860697	1.54321	0.994109	0.7004831
0.7	5.851258	1.788354	0.9728561	0.6281442	0.4422378
0.9	1.994876	0.6203595	0.3394195	0.2191803	0.1539430
$\pi_y = 0$					
0.1	15.14621	3.881178	1.912665	1.169676	0.8013822
0.3	12.01967	3.228195	1.670384	1.072838	0.7717918
0.5	8.768267	2.461209	1.331558	0.8938917	0.6719963
0.7	5.378236	1.573779	0.8873309	0.6206652	0.4864873
0.9	1.834554	0.5583768	0.3272584	0.2380587	0.1944700

Table 6: *PRE of the estimator* $\hat{\pi}_{mm}$ *with respect to* $\hat{\pi}_b$ *for the whole range of* π *and* π_y, *when* k_1=1.1, 1.3, 1.5, 1.7 *and* 1.9, *and* n=30.

$\pi_y = 0.1$					
	k_1				
π	1.1	1.3	1.5	1.7	1.9
0.1	8.743155	7.330011	6.487413	5.933581	5.546242
0.3	7.870878	6.735641	6.103625	5.724755	5.493355
0.5	7.022722	6.145025	5.712068	5.507161	5.442771
0.7	6.189962	5.553399	5.309261	5.27805	5.39474
0.9	5.364007	4.955686	4.891103	5.033898	5.349634
$\pi_y = 0.3$					
0.1	36.47786	31.16546	27.97289	25.88430	24.44566
0.3	34.90854	30.20559	27.78055	26.51907	25.96383
0.5	33.68503	29.49135	27.86026	27.58453	28.25562
0.7	32.83734	29.0609	28.31008	29.32173	31.91255
0.9	32.45881	28.99828	29.31674	32.22184	38.35979
$\pi_y = 0.7$					
0.1	38.82638	34.08307	30.71341	28.1714	26.16838
0.3	39.60037	33.5545	29.16792	25.82964	23.19810
0.5	41.45413	33.41943	27.81298	23.72201	20.6262
0.7	45.24942	33.81432	26.6302	21.79475	18.35848
0.9	53.37514	35.02314	25.60984	20.00232	16.32461
$\pi_y = 0.9$					
0.1	9.392958	8.568984	7.943635	7.447896	7.041731
0.3	8.493113	7.725001	7.083384	6.539587	6.072944
0.5	7.613704	6.916177	6.284498	5.727684	5.24215
0.7	6.745091	6.134519	5.536745	4.994966	4.519994
0.9	5.877475	5.372653	4.831529	4.327892	3.884813
$\pi_y = 1$					
0.1	6.056935	5.603668	5.247049	4.956157	4.712218
0.3	5.454545	5.035247	4.66497	4.338395	4.049844
0.5	4.863222	4.488778	4.123711	3.786575	3.483309
0.7	4.278729	3.960396	3.617363	3.289474	2.992383
0.9	3.696858	3.446578	3.14095	2.838221	2.562132
$\pi_y = 0$					
0.1	5.55016	4.602992	4.046194	3.682980	3.429901
0.3	4.993758	4.226096	3.800639	3.543743	3.383434
0.5	4.447533	3.848210	3.547672	3.396372	3.334491
0.7	3.907886	3.467331	3.285571	3.239129	3.282357
0.9	3.37126	3.081325	3.012332	3.069839	3.226067

Table 7: *PRE of the estimator $\hat{\pi}_{mm}$ with respect to $\hat{\pi}_b$ for the whole range of π and π_y, when k_1 =1.1, 1.3, 1.5, 1.7 and 1.9, and n =50.*

$\pi_y = 0.1$					
			k_1		
π	1.1	1.3	1.5	1.7	1.9
0.1	5.250182	4.400747	3.894415	3.561667	3.328977
0.3	4.729157	4.045753	3.665436	3.437482	3.298247
0.5	4.21612	3.688707	3.428557	3.305408	3.266658
0.7	3.70788	3.327744	3.182066	3.163725	3.233895
0.9	3.201259	2.960854	2.923952	3.010252	3.199526
$\pi_y = 0.3$					
0.1	21.68695	18.56644	16.68551	15.45289	14.60300
0.3	20.25565	17.65535	16.30691	15.60873	15.30996
0.5	18.88867	16.78058	15.97612	15.88828	16.31467
0.7	17.55495	15.92775	15.69837	16.34340	17.80085
0.9	16.21790	15.07965	15.48342	17.06479	20.1466
$\pi_y = 0.7$					
0.1	22.80040	20.10128	18.16781	16.70031	15.53848
0.3	22.01745	18.98662	16.70396	14.92005	13.48594
0.5	21.3365	17.92313	15.32463	13.31486	11.73084
0.7	20.76399	16.89182	14.00643	11.84831	10.20482
0.9	20.31833	15.87086	12.72710	10.49122	8.858181
$\pi_y = 0.9$					
0.1	5.631635	5.138258	4.763713	4.466735	4.223376
0.3	5.078863	4.622332	4.240342	3.916180	3.637737
0.5	4.534076	4.124624	3.75204	3.422508	3.134455
0.7	3.993647	3.641953	3.293924	2.976347	2.696618
0.9	3.453932	3.171395	2.861854	2.570231	2.311631
$\pi_y = 1$					
0.1	3.633721	3.361858	3.147954	2.973468	2.827141
0.3	3.269161	3.018412	2.796847	2.601344	2.428545
0.5	2.909091	2.686567	2.469136	2.268041	2.086957
0.7	2.551950	2.364765	2.161902	1.967342	1.790656
0.9	2.196193	2.051583	1.872659	1.694277	1.530925
$\pi_y = 0$					
0.1	3.3325	2.763298	2.428781	2.210602	2.058596
0.3	3.00075	2.538554	2.282513	2.127943	2.031488
0.5	2.672234	2.311396	2.130493	2.039397	2.002086
0.7	2.345648	2.08103	1.971846	1.943918	1.969826
0.9	2.019692	1.846600	1.805563	1.840209	1.933955