A New Simple Proof of the No-arbitrage Theorem for Multi-period Binomial Model

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Abstract

Binomial option pricing model is one of the widely used models to price option contracts, which are commonly employed to hedge against risks in the insurance field. One of the main underlying assumptions of the binomial model is the no-arbitrage condition. This condition simply provides a necessary and sufficient condition for the model to be free of arbitrage opportunities. Previous attempts were made to assess the viability of this assumption. But they were quite complicated and lengthy which tend to obscure the underlying meaning, and may appear to be daunting and inaccessible to the general audience. In this paper, we supply a clear, clean and simple proof using only pre-trigonometric algebra.

1. Introduction

Insurance companies often rely on investment opportunities outside the traditional insurance sector to manage their financial risk. In fact, the payoffs of many poplar equity-linked insurance contracts such as variable annuity (VA) contracts, segregated fund contracts and equity-indexed annuity contracts can be expressed in terms of options. (c.f. Hardy (2003)) Many option pricing models have been proposed and successfully used in quantitative finance over the past decades. (c.f. Hull (2009)) and Karatzas and Shreve (2005))

The binomial model is a well-known asset pricing model first introduced by Cox, Ross and Rubinstein (1979). It is widely used for option pricing. (c.f. Hull (2009) or Shreve (2005)). Its simplicity makes it easy to apply in many cases. For European options, the model usually provides a closed-form solution. (c.f. Brealey, Myers and Allen (2005), Copeland, Weston and Shastri (2005) and Hull (2009)) For American options, a closed form solution is usually not available, but one can work backward and value it analytically with the aid of a computer program. (c.f. Geske and Johnson (1984)) Moreover, the model is important in its own right as the famous Black-Scholes-Merton formula can be derived as a limit of the binomial model. (c.f. Cox, Ross and Rubinstein (1979) and Shreve (2005)) In addition, the binomial tree model can be extended to a trinomial tree model for pricing exotic options with regime-switching. (c.f. Yue and Yanc (2010)) Besides its application in financial option, the binomial model has also been applied to evaluate real options. (c.f. Copeland, Weston and Shastri (2005) and Trigeorgis (1996)) Thus, a thorough understanding of the binomial model is essentially for theorists, empiricists and practitioners in economics, finance and actuarial science. In the following, we mainly adopt the notations in Bjork (2004).

The multi-period binomial model can be described as follows. Assume there exists a bond B at time t, with the price $B=(1+r)^t$, where $t=0,1,2,\ldots$. We also assume there exists a stock, with the price S_t, which is a random variable. We consider time $t=0$ as the beginning of the period. Thus, $B_0=1$ and S_0 are the initial prices of the bond and the stock respectively.

(We assume S_0 is a positive constant.) At the end of the first period, the stock price will either move from S_0 up to uS_0 with up probability p_u, or go from S_0 down to dS_0 with down probability p_d, where $0 < d < u$ and $p_d + p_u = 1$. Thus, we have

$$S_1 = \begin{cases} uS_0, \text{ with probability } p_u, \\ dS_0, \text{ with probability } p_d. \end{cases}$$

Similarly, at the end of the second period, the stock price will either move from S_1 up to uS_1 with up probability p_u or go from S_1 down to dS_1 with down probability p_d. Hence,

$$S_2 = \begin{cases} uS_1, \text{ with probability } p_u, \\ dS_1, \text{ with probability } p_d. \end{cases}$$

This process continues with the up probability p_u and the down probability p_d unchanged. Therefore, S_2 can be expressed as

$$S_2 = S_0 Z_0 Z_1$$

Z_0 and Z_1 are two i.i.d. binomial random variables. Following this line of reasoning, we may simply write the price of the stock at the end of t-th period ($t \geq 1$) as

$$S_t = S_0 Z_0 Z_1 ... Z_{t-1}$$

where $S_0 > 0$ is a constant, and $Z_0, Z_1, Z_2 ...$ are i.i.d. binomial random variables such that

$$Z_k = \begin{cases} u, \text{ with probability } p_u \\ d, \text{ with probability } p_d \end{cases} \qquad (1.1)$$

Definition 1.1 A *portfolio* $h_t = (x_t, y_t)$ represents the number of bonds and shares at time $t \geq 1$. We assume that h_t is a measurable function of $S_0, S_1, ..., S_{t-1}$

$$h_t = h_t(S_0, S_1 ..., S_{t-1}),$$

with the convention that $h_0 = h_1$. From advanced probability (c.f. Billingsley (1995), Chow and Teicher (1997) or Shiryaev (1984)), we know h_t itself is a random variable.

Definition 1.2 The *value* of the portfolio at time t is denoted by V_t^h. Assuming compound interest with constant effective rate r, we then have

$$V_t^h = x_t B_t + y_t S_t = x_t (1+r)^t + y_t S_0 Z_0 ... Z_{t-1}. \qquad (1.2)$$

To state the no-arbitrage condition theorem, we need to introduce the following concept:

Definition 1.3 A portfolio is called *self-financing* if the following identity holds for all $t \geq 0$:

$$x_t (1+r)^t + y_t S_t = x_{t+1} (1+r)^t + y_{t+1} S_t. \qquad (1.3)$$

Remark. Recall $B = (1+r)^t$. Equation (1.3) means no external financing is needed at any time. The portfolio, once formed, can fund itself. If we let

$$z_t = x_t (1+r)^{t-1}, \ t \geq 1 \qquad (1.4)$$

then equation (1.3) can be written as

$$z_t (1+r) + y_t S_t = z_{t+1} + y_{t+1} S_t. \qquad (1.5)$$

Note that Bjork (2004) uses (1.5) to define self-financing.

Definition 1.4 An *arbitrage possibility* is a self-financing portfolio such that

$$V_0^h = 0,$$
$$P(V_T^h \geq 0) = 1,$$
$$P(V_T^h \geq 0) > 0.$$

With all these concepts, we now can state the no-arbitrage condition theorem.

Theorem 1 (No-arbitrage Condition Theorem)

The multi-period binomial model is arbitrage-free if and only if

$$0 < d < 1 + r < u \qquad (1.6)$$

Shreve (2005) shows the necessity in the one-period model using trading strategy. Bjork (2004) gives a lengthy proof for the multi-period model using martingale probability and binomial algorithm. In the next section, we provide a new and simple proof using only elementary algebra.

2. A New and Simple Proof of No-arbitrage Condition Theorem

Since $V_0^h = 0$, equation (1.5) implies

$$z_1 = x - S_0 y_1 . \qquad (2.1)$$

From (1.5), we also have that

$$z_{t+1} = z_t(1+r) + y_t S_t - y_{t+1} S_t . \qquad (2.2)$$

First, let's consider the case $T = 1$. In this case,

$$V_T^h = V_1^h = \begin{cases} x_1(1+r) + y_1 u S_0, \text{ with probability } p_u, \\ x_1(1+r) + y_1 d S_0, \text{ with probability } p_d. \end{cases}$$

Now by (1.4) and (2.1), we can write V_1^h as

$$V_T^h = V_1^h = \begin{cases} -y_1[u-(1+r)]S_0, \text{ with probability } p_u, \\ -y_1[d-(1+r)]S_0, \text{ with probability } p_d. \end{cases}$$

Multiplying the above equalities together, we see

$$-y_1[u-(1+r)]S_0 < 0 \text{ or } -y_1[d-(1+r)]S_0 < 0$$

if and only if

$$(y_1 S_0)^2 [u-(1+r)][d-(1+r)] < 0 \qquad (2.3)$$

However, (2.3) holds if and only if

$$d < 1 + r < u$$

Since we assume $d > 0$, $p_u > 0$ and $p_d > 0$, this shows there exist no arbitrage opportunities if and only if $0 < d < 1 + r < u$.

Now let's consider the case $T = 2$. By (2.1) and (2.2), we have

$$z_2 = -y_1(1+r)S_0 + y_1(1+r)S_0 + y_1 Z_1 S_0 - y_2 Z_1 S_0 . \qquad (2.4)$$

Substituting (2.4) into (1.2), we obtain

$$V_2^h = -y_1(1+r)^2 S_0 + y_1(1+r)Z_1 S_0 - (1+r)y_2 Z_1 + y_2 Z_1 Z_2 S_0 > 0.$$

In view of (1.1), if there exists an arbitrage opportunity h_t, then the following inequalities hold:

$$-y_1(1+r)^2 + y_1(1+r)u - (1+r)y_2u + y_2u^2 > 0,$$
$$-y_1(1+r)^2 + y_1(1+r)u - (1+r)y_2u + y_2ud > 0,$$
$$-y_1(1+r)^2 + y_1(1+r)d - (1+r)y_2u + y_2ud > 0,$$
$$-y_1(1+r)^2 + y_1(1+r)d - (1+r)y_2u + y_2d^2 > 0.$$

Put $U = u-(1+r)$ and $D = d-(1+r)$. Then the above inequalities can be written as

$$\begin{aligned} &y_1(1+r)U + y_2uU > 0,\\ &y_1(1+r)U + y_2uD > 0,\\ &y_1(1+r)D + y_2dU > 0,\\ &y_1(1+r)D + y_2dD > 0. \end{aligned} \qquad (2.5)$$

Note that the left-hand sides of these inequalities are the direct product of D and U with $(1+r,u)$ and $(1+r,d)$. Suppose (1.6) holds, then

$$D < 0 < U$$

and hence the direct product of D and U with $(1+r,u)$ and $(1+r,d)$ cannot all be positive. In other words, (2.5) cannot hold. This contradicts our assumption that h_t is an arbitrage opportunity. Thus, sufficiency is proved.

On the other hand, if (1.6) fails to hold, say,

$$0 < d < u < (1+r),$$

i.e.

$$D < U < 0,$$

then any portfolio with $y_t < 0$ is clearly an arbitrage opportunity. Similarly, if

$$0 < (1+r) < d < u,$$

i.e.

$$0 < D < U,$$

then any portfolio with $y_t > 0$ will be an arbitrage opportunity. Thus, the necessity is established too. This completes the proof for the case $T = 2$. The general case can be shown by following exactly the same line of reasoning used for the case $T = 2$ but in the space of higher dimensions.

3. Summary

The necessary and sufficient condition for the multi-period binomial model to be arbitrage-free can be clearly stated in terms of the up factor, the down factor and the interest rate. The formula is simple. However, most proofs in the existing literature involve advanced mathematics such as martingale theory. Here we provide a proof which only requires elementary algebra. Hence, our argument is completely accessible to the general audience. The proof assumes a constant force of interest. Relaxing this assumption can lead to further research. In that case, to characterize the condition for the model to be arbitrage-free can be challenging. One probably has to resort to heavy probabilistic machineries such as martingales. However, the proof in this paper will offer some insight.

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